Long-range memory model of trading activity and volatility

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Abstract. Earlier we proposed the stochastic point process model, which reproduces a variety of self-affine time series exhibiting power spectral density $S(f)$ scaling as power of the frequency f and derived a stochastic differential equation with the same long range memory properties. Here we present a stochastic differential equation as a dynamical model of the observed memory in the financial time series. The continuous stochastic process reproduces the statistical properties of the trading activity and serves as a background model for the modeling waiting time, return and volatility. Empirically observed statistical properties: exponents of the power-law probability distributions and power spectral density of the long-range memory financial variables are reproduced with the same values of few model parameters.

1. Introduction

Stochastic volatility models of the financial time series are fundamental to investment, option pricing and risk management [1, 2]. The volatility serves as a quantitative price diffusion measure in widely accepted stochastic multiplicative process known as geometric Brownian motion (GBM). Extensive empirical data analysis of big price movements in the financial markets confirms the assumption that volatility itself is a stochastic variable or more generally the function of a stochastic variable [3]. By analogy with physics we can assume that speculative prices $p(t)$ change in a "random medium" described by the random diffusion coefficient. Such analogy may be reversible from the point that the complex models of stochastic price movements can be applicable for the description of complex physical systems such as stochastic resonance, noise induced phase transitions and high energy physics applications. This analogy contributes to further development of statistical mechanics - nonextensive one and superstatistics have been introduced [4, 5].

Additive-multiplicative stochastic models of the financial mean-reverting processes provide rich spectrum of shapes for the probability distribution function (PDF) depending on the model parameters [6]. Such stochastic processes model the empirical PDF's of volatility, volume and price returns with success when the appropriate fitting parameters are selected. Nevertheless, there is the necessity to select the most appropriate stochastic models able to describe volatility as well as other variables under the dynamical aspects and the long range correlation aspects. There is empirical evidence that trading activity, trading volume, and volatility are stochastic variables with the long-range correlation [1, 7, 8] and this key aspect is not accounted for in widespread models. Moreover, rather often there is evidence that the models proposed are characterized only by the short-range time memory [9].

Phenomenological descriptions of volatility, known as heteroscedasticity, have proven to be of extreme importance in the option price modeling [10]. Autoregressive conditional heteroscedasticity (ARCH) processes and more sophisticated structures GARCH are proposed as linear dependencies on previous values of squared returns and variances [10, 11]. These models based on empirically fitted parameters fail in reproducing power law behavior of the volatility autocorrelation function. We do believe that the stochastic models with the limited number of parameters and minimum stochastic variables are possible and would better reflect the market dynamics and its response to the external noise.

Recently we investigated analytically and numerically the properties of stochastic multiplicative point processes [12], derived formula for the power spectrum and related the model with the general form of multiplicative stochastic differential equation [13]. Preliminary the comparison of the model with the empirical data of spectrum and probability distribution of stock market trading activity [14] stimulated us to work on the definition of more detailed model. The extensive empirical analysis of the financial market data, supporting the idea that the long-range volatility correlations

arise from trading activity, provides valuable background for further development of the long-ranged memory stochastic models [7, 8]. We will present the stochastic model of trading activity with the long-range correlation and will investigate its' connection to the stochastic modeling of volatility and returns.

2. Stochastic model of interevent time

Earlier we proposed the stochastic point process model, which reproduced a variety of self-affine time series exhibiting the power spectral density $S(f)$ scaling as power of the frequency f [12, 14]. The time interval between point events in this model fluctuates as a stochastic variable described by the multiplicative iteration equation

$$\tau_{k+1} = \tau_k + \gamma\tau_k^{2\mu-1} + \sigma\tau_k^{\mu}\varepsilon_k. \tag{1}$$

Here interevent time $\tau_k = t_{k+1} - t_k$ between subsequent events k and $k+1$ fluctuates due to the random perturbation by a sequence of uncorrelated normally distributed random variable $\{\varepsilon_k\}$ with the zero expectation and unit variance, σ denotes the standard deviation of the white noise and $\gamma \ll 1$ is a coefficient of the nonlinear damping. It has been shown analytically and numerically [12, 14] that the point process with stochastic interevent time (1) may generate signals with the power-law distributions of the signal intensity and $1/f^{\beta}$ noise. The corresponding Ito stochastic differential equation for the variable $\tau(t)$ as a function of the actual time can be written as

$$d\tau = \gamma\tau^{2\mu-2}dt + \sigma\tau^{\mu-1/2}dW, \tag{2}$$

where W is a standard random Wiener process. Eq. (2) describes the continuous stochastic variable $\tau(t)$ which can be assumed as slowly diffusing mean interevent time of Poisson process with the stochastic rate $1/\tau(t)$. We put the modulated Poisson process into the background of the long-range memory point process model.

The diffusion of τ must be restricted at least from the side of high values. Therefore we introduce a new term $-\frac{m}{2}\sigma^2\left(\frac{\tau}{\tau_0}\right)^m\tau^{2\mu-2}$ into the Eq. (2), which produces the exponential diffusion reversion in equation

$$d\tau = \left[\gamma - \frac{m}{2}\sigma^2\left(\frac{\tau}{\tau_0}\right)^m\right]\tau^{2\mu-2}dt + \sigma\tau^{\mu-1/2}dW, \tag{3}$$

where m and τ_0 are the power and value of the diffusion reversion, respectively. The associated Fokker-Plank equation with the zero flow will give the simple stationary PDF

$$P(\tau) \sim \tau^{\alpha+1}\exp\left[-\left(\frac{\tau}{\tau_0}\right)^m\right] \tag{4}$$

with $\alpha = 2(\gamma_{\sigma}-\mu)$, where $\gamma_{\sigma} = \gamma/\sigma^2$. We define the conditional probability of interevent time τ_{p} in the modulated Poisson point process with stochastic rate $1/\tau$ as

$$\varphi(\tau_{\mathrm{p}}|\tau) = \frac{1}{\tau}\exp\left[-\frac{\tau_{\mathrm{p}}}{\tau}\right]. \tag{5}$$

Then the long time distribution $\varphi(\tau_{\mathrm{p}})$ of interevent time τ_{p} has the integral form

$$\varphi(\tau_{\mathrm{p}}) = C\int_0^{\infty}\exp\left[-\frac{\tau_{\mathrm{p}}}{\tau}\right]\tau^{\alpha}\exp\left[-\left(\frac{\tau}{\tau_0}\right)^m\right]d\tau, \tag{6}$$

with C defined from the normalization, $\int_0^\infty \varphi(\tau_\text{p})\text{d}\tau_\text{p} = 1$. In the case of pure exponential diffusion reversion, $m = 1$, PDF (6) has a simple form

$$\varphi(\tau_\text{p}) = \frac{2}{\Gamma(2+\alpha)\tau_0} \left(\frac{\tau_\text{p}}{\tau_0}\right)^{\frac{1+\alpha}{2}} \text{K}_{(1+\alpha)} \left(2\sqrt{\frac{\tau_\text{p}}{\tau_0}}\right), \tag{7}$$

where $\text{K}_\alpha(z)$ denotes the modified Bessel function of the second kind. For $m > 1$ more complicated structures of distribution $\varphi(\tau_\text{p})$ expressed in terms of hypergeometric functions arise.

3. Stochastic model of flow of points or events

The introduced stochastic multiplicative model of interevent time, the interval between trades in the financial market, defines the model of event flow n. First of all we apply Ito transformation of variables introducing flow of events $n(t) = \frac{1}{\tau(t)}$. The stochastic differential equation for n follows from Eq. (2),

$$\text{d}n = \sigma^2 \left[(1 - \gamma_\sigma) + \frac{m}{2}\left(\frac{n_0}{n}\right)^m\right] n^{2\eta-1}\text{d}t + \sigma n^\eta \text{d}W, \tag{8}$$

where $\eta = \frac{5}{2} - \mu$ and $n_0 = 1/\tau_0$. Eq. (8) describes stochastic process n with PDF

$$P(n) \sim \frac{1}{n^\lambda} \exp\left\{-\left(\frac{n_0}{n}\right)^m\right\}, \quad \lambda = 2(\eta - 1 + \gamma_\sigma), \tag{9}$$

and power spectrum $S(f)$ [12, 13, 14]

$$S(f) \sim \frac{1}{f^\beta}, \quad \beta = 2 - \frac{3 - 2\gamma_\sigma}{2\eta - 2}. \tag{10}$$

Noteworthy, that in the proposed model only two parameters, γ_σ and η (or μ), define exponents λ and β of two power-law statistics, i.e. of PDF and power spectrum. Time scaling parameter σ^2 in Eq. (8) can be omitted adjusting the time scale.

Stochastic variable n denotes the number of events per unit time interval. One has to integrate the stochastic signal Eq. (8) in the time interval τ_d to get number of events in the selected time window. In this paper we will denote the integrated number of points or events as

$$N(t, \tau_\text{d}) = \int_t^{t+\tau_\text{d}} n(t')\text{d}t' \tag{11}$$

and will call it trading activity in the case of the financial market. Flow of points or events arises in different fields, such as physics, economics, cosmology, ecology, neurology, the Internet, seismology, i.e., electrons, photons, cars, pulses, events, and so on, or subsequent actions, like seismic events, neural action potentials, transactions in the financial markets, human heart beats, biological ion-channel openings, burst errors in many communication systems, the Internet network packets, etc. We will discuss possible application of the proposed stochastic model to model the trading activity in the financial markets.

4. Stochastic model of trading activity

It is widely accepted that in high-frequency financial data not only the returns but also the waiting times between the consecutive trades are random variables [15]. Waiting times between trades do not follow the exponential distribution and the related point process is not the Poisson one. The extensive empirical analysis provides evidence that the related stochastic variable trading activity defined as flow of trades is stochastic variable with the long range memory [16]. We will investigate how the proposed modulated Poisson stochastic point process can be adjusted to model trading activity with the empirically defined statistical properties. Detrended fluctuation analysis [16] is one of the methods to define the second order statistics, the autocorrelation of trading activity. The histogram of the detrended fluctuation analysis exponents ν obtained by fits for each of the 1000 US stocks shows a relatively narrow spread of ν around the mean value $\nu = 0.85 \pm 0.01$ [16]. We use relation between the exponents of detrended fluctuation analysis and the exponents of power spectrum $\beta = 2\nu - 1$ [17] and in this way define the empirical value of the exponent for the power spectral density $\beta = 0.7$. Our analysis of the Lithuanian stock exchange data confirmed that the power spectrum of trading activity is the same for various liquid stocks even for the emerging markets [18]. The histogram of exponents obtained by fits to the cumulative distributions of trading activites of 1000 US stocks [16] gives the value of exponent $\lambda = 4.4 \pm 0.05$ describing the power-law behavior of the trading activity. Empirical values of $\beta = 0.7$ and $\lambda = 4.4$ confirm that the time series of the trading activity in real markets are fractal with the power law statistics. Time series generated by stochastic process (8) are fractal in the same sense.

Nevertheless, we face serious complications trying to adjust model parameters to the empirical data of the financial markets. For the pure multiplicative model, when $\mu = 1$ or $\eta = 3/2$, we have to take $\gamma_\sigma = 0.85$ to get $\beta = 0.7$ and $\gamma_\sigma = 1.7$ to get $\lambda = 4.4$, i.e. it is impossible to reproduce the empirical PDF and power spectrum with the same relaxation parameter γ_σ and exponent of multiplicativity μ. We have proposed possible solution of this problem in our previous publications [14, 18] deriving PDF for the trading activity N

$$P(N) \sim \begin{cases} \frac{1}{N^{3+\alpha}}, & N \ll \gamma^{-1}, \\ \frac{1}{N^{5+2\alpha}}, & N \gg \gamma^{-1}. \end{cases} \tag{12}$$

When $N \gg \gamma^{-1}$ this yields exactly the required value of $\lambda = 5 + 2\alpha = 4.4$ and $\beta = 0.7$ for $\gamma_\sigma = 0.85$.

Nevertheless, we cannot accept this as the sufficiently accurate model of the trading activity as the empirical power law distribution is achieved only for very high values of the trading activity. Probably this reveals the mechanism how the power law distribution converges to normal distribution through the growing values of the exponent, but empirically observed power law distribution in wide area of N values cannot be reproduced. Let us notice here that the desirable power law distribution of the trading activity with the exponent $\lambda = 4.4$ may be generated by the model (8)
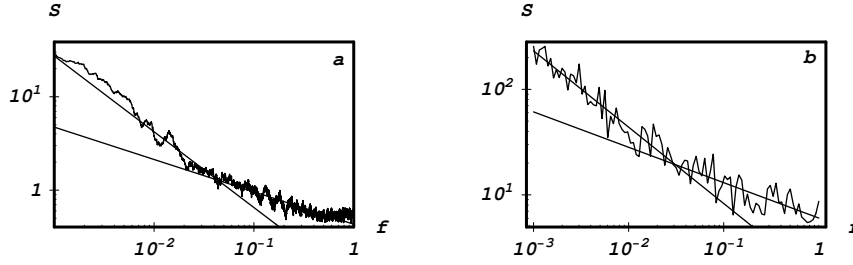
Figure 1. Power spectral density $S(f)$ calculated with parameters $\gamma = 0.0004$; $\sigma = 0.025$; $\epsilon = 0.07$; $\tau_0 = 1$; $m = 6$. Straight lines approximate power spectrum $S \sim 1/f^{\beta_{1,2}}$ with $\beta_1 = 0.33$ and $\beta_2 = 0.72$: a) $S(f)$ calculated by the Fast Fourier Transform of n series generated by Eq. (13), b) $S(f)$ averaged over 20 series of 100000 iterations of the flow $I(t) = \sum_k \delta(t - t_k)$ with the interevent time $\tau_k = t_{k+1} - t_k$ generated by Eq. (14).

with $\eta = 5/2$ and $\gamma_\sigma = 0.7$. Moreover, only the smallest values of τ or high values of n contribute to the power spectral density of trading activity [13]. This suggests us to combine the point process with two values of μ: (i) $\mu \simeq 0$ for the main area of diffusing τ and n and (ii) $\mu = 1$ for the lowest values of τ or highest values of n. Therefore, we introduce a new stochastic differential equation for n combining two powers of multiplicative noise,

$$\mathrm{d}n = \sigma^2 \left[(1 - \gamma_\sigma) + \frac{m}{2} \left(\frac{n_0}{n} \right)^m \right] \frac{n^4}{(n\epsilon + 1)^2} \mathrm{d}t + \frac{\sigma n^{5/2}}{(n\epsilon + 1)} \mathrm{d}W, \tag{13}$$

where a new parameter ϵ defines crossover between two areas of n diffusion. The corresponding iterative equation of form (1) for τ_k in such a case is

$$\tau_{k+1} = \tau_k + \left[\gamma - \frac{m}{2}\sigma^2 \left(\frac{\tau}{\tau_0} \right)^m \right] \frac{\tau_k}{(\epsilon + \tau_k)^2} + \sigma \frac{\tau_k}{\epsilon + \tau_k} \varepsilon_k. \tag{14}$$

Eqs. (13) and (14) define related stochastic variables $n = \frac{1}{\tau}$ and τ, respectively, and they should reproduce the long range statistical properties of the trading activity and of waiting time in the financial markets. We verify this by the numerical calculations. In figure 1 we present the power spectral density calculated for the equivalent processes (13) and (14) (see [14] for details of calculations). This approach reveals the structure of the power spectral density in wide range of frequencies and shows that the model exhibits not one but rather two separate power laws with the exponents $\beta_1 = 0.33$ and $\beta_2 = 0.72$. From many numerical calculations performed with the multiplicative point processes we can conclude that combination of two power laws of spectral density arise only when multiplicative noise is a crossover of two power laws, see (13) and (14). We will show in the next section that this may serve as an explanation of two exponents of the power spectrum in the empirical data of volatility for S&P 500 companies [19].

Empirical data of the trading activity statistics must be modeled by the integrated flow of event N defined in the time interval $\tau_\mathrm{d} \gg \tau_0$. In figure 2 we demonstrate the cumulative probability distribution functions $P_>(n)$ calculated from the histogram of
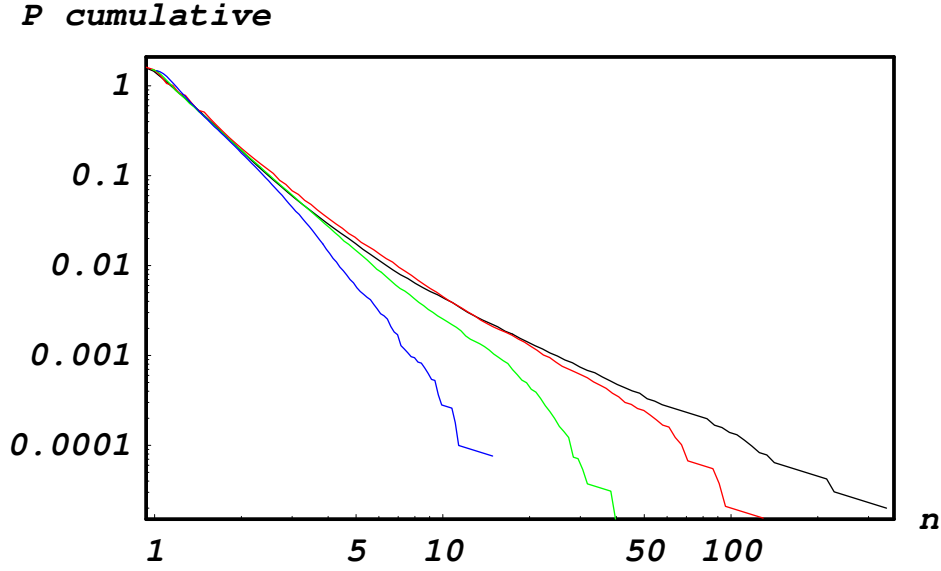
P cumulative

Figure 2. Cumulative PDF $P_>(n)$ calculated from the histogram of N/τ_{d} generated by Eq. (14) with the increasing time interval τ_{d} from the above curve: $\tau_{\mathrm{d}} = 1, 10, 50$ and 250. Other parameters are as in figure 1.

N/τ_{d} generated by Eq. (14) with increasing time interval τ_{d}. This illustrates how distribution of the integrated signal N converges to the normal distribution (the central limit theorem) through growing exponent of the power-law distribution and provides an evidence that the empirically observed exponent $\lambda = 4.4$ of the power-law distribution of N [7, 8] can be explained by the proposed model with the same parameters suitable for description of the power spectrum of the trading activity.

The power spectrum of the trading activity N can be calculated by the Fast Fourier Transform of the generated numerical series. As illustrated in figure 3, the exponents $\beta = 0.7$ of the power spectrum are independent of τ_{d} and reproduce the empirical results of the detrended fluctuation analysis [7, 8].

The same numerical results can be reproduced by continuous stochastic differential equation (13) or iteration equation (14). One can consider the discrete iterative equation for the interevent time τ_k (14) as a method to solve numerically continuous equation

$$\mathrm{d}\tau = \left[\gamma - \frac{m}{2}\sigma^2\left(\frac{\tau}{\tau_0}\right)^m\right]\frac{1}{(\epsilon + \tau)^2}\mathrm{d}t + \sigma\frac{\sqrt{\tau}}{\epsilon + \tau}\mathrm{d}W. \tag{15}$$

The continuous equation (13) follows from the Eq. (15) after change of variables $n = 1/\tau$.

We can conclude that the long range memory properties of the trading activity in the financial markets as well as the PDF can be modeled by the continuous stochastic differential equation (13). In this model the exponents of the power spectral density, β, and of PDF, λ, are defined by one parameter $\gamma_\sigma = \gamma/\sigma^2$. We consider the continuous equation of the mean interevent time τ as a model of slowly varying stochastic rate $1/\tau$
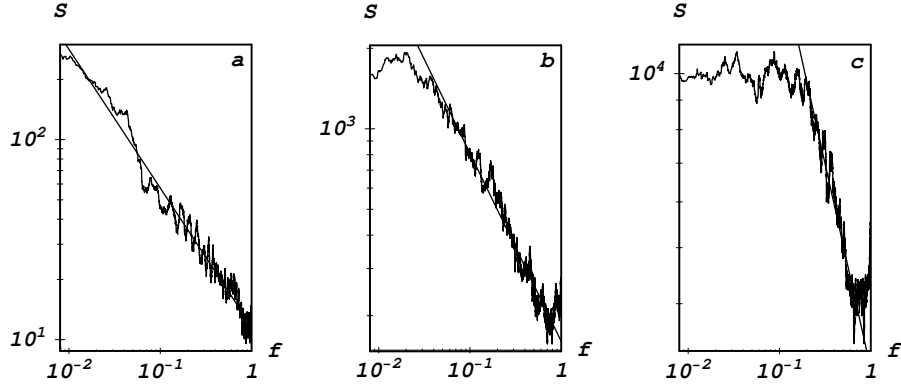
Figure 3. Power spectral density of the trading activity N calculated by the Fast Fourier Transform of N series generated with Eq. (13) for the same parameters as in figures 1 and 2: a) $\tau_d = 10$; b) $\tau_d = 50$; c) $\tau_d = 250$. Straight lines approximate power spectrum $S \sim 1/f^\beta$, with $\beta = 0.7$.
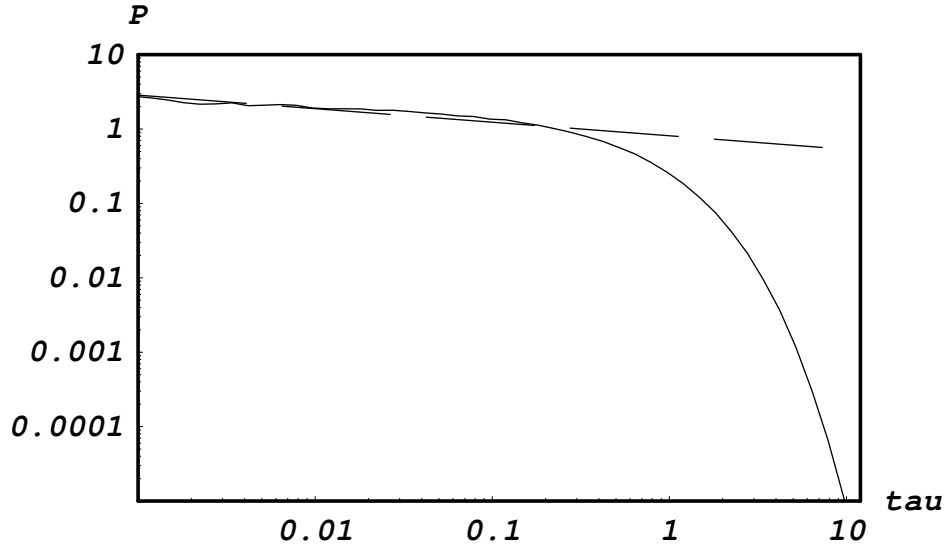

Figure 4. Probability distribution function $P(\tau_p)$ calculated from the histogram of τ_p generated by Eq. (16) with rate calculated from Eq. (15). Used parameters are $\gamma = 0.0004$; $\sigma = 0.025$; $\epsilon = 0.07$; $\tau_0 = 1$ and $m = 6$. Dashed line approximates power law $P(\tau_p) \sim \tau_p^{-0.15}$.

in the modulated Poisson process

$$\varphi(\tau_p|\tau) = \frac{1}{\tau} \exp\left[-\frac{\tau_p}{\tau}\right]. \tag{16}$$

In figure 4 we demonstrate the probability distribution functions $P(\tau_p)$ calculated from the histogram of τ_p generated by Eq. (16) with the diffusing mean interevent time calculated from Eq. (15).

Numerical results show good qualitative agreement with the empirical data of interevent time probability distribution measured from few years series of U.S. stock data [20]. This enables us to conclude that the proposed stochastic model captures the main statistical properties including PDF and the long range correlation of the trading activity in the financial markets. Furthermore, in the next section we will show that this may serve as a background statistical model responsible for the statistics of return volatility in widely accepted GBM of the financial asset prices.

5. Modeling returns and volatility

We follow an approach developed in [7, 8, 16] to analyze the empirical data of price fluctuations driven by the market activity. The basic quantities studied for the individual stocks are price $p(t)$ and return

$$x(t, \tau_{\mathrm{d}}) = \ln p(t + \tau_{\mathrm{d}}) - \ln p(t) \tag{17}$$

Return $x(t, \tau_{\mathrm{d}})$ over a time interval τ_{d} can be expressed through the subsequent changes δx_i due to the trades $i = 1, 2....N(t, \tau_{\mathrm{d}})$ in the time interval $[t, t + \tau_{\mathrm{d}}]$,

$$x(t, \tau_{\mathrm{d}}) = \sum_{i=1}^{N(t,\tau_{\mathrm{d}})} \delta x_i. \tag{18}$$

We denote the variance of δx_i calculated over the time interval τ_{d} as $W^2(t, \tau_{\mathrm{d}})$. If δx_i are mutually independent one can apply the central limit theorem to sum (18). This implies that for the fixed variance $W^2(t, \tau_{\mathrm{d}})$ return $x(t, \tau_{\mathrm{d}})$ is a normally distributed random variable with the variance $W^2(t, \tau_{\mathrm{d}})N(t, \tau_{\mathrm{d}})$

$$x(t, \tau_{\mathrm{d}}) = W(t, \tau_{\mathrm{d}})\sqrt{N(t, \tau_{\mathrm{d}})}\varepsilon_t. \tag{19}$$

Empirical test of conditional probability $P(x(t, \tau_{\mathrm{d}})|W(t, \tau_{\mathrm{d}}))$ [7] confirms its' Gaussian form, and the unconditional distribution $P(x(t, \tau_{\mathrm{d}}))$ is a power-law with the cumulative exponent 3. This implies that the power-law tails of returns are largely due to those of $W(t, \tau_{\mathrm{d}})$. Here we refer to the theory of price diffusion as a mechanistic random process [21, 22]. For this idealized model the short term price diffusion depends on the limit order removal and this way is related to the market order flow. Furthermore, the empirical analysis confirms that the volatility calculated for the fixed number of transactions has the long memory properties as well and it is correlated with real time volatility [23]. We accumulate all these results into strong assumption that standard deviation $W(t, \tau_{\mathrm{d}})$ may be proportional to the square root of the trading activity, i.e., $W(t, \tau_{\mathrm{d}}) = k\sqrt{N(t, \tau_{\mathrm{d}})}$. This enables us to propose a very simple model of return

$$x(t, \tau_{\mathrm{d}}) = kN(t, \tau_{\mathrm{d}})\varepsilon_t \tag{20}$$

and related model of volatility $v = |x(t, \tau_{\mathrm{d}})|$ based on the proposed model of trading activity (13). We generate series of trade flow $n(t)$ numerically solving Eq. (13) with variable steps of time $\Delta t_i = h_i = n_0/n_i$ and calculate the trading activity in
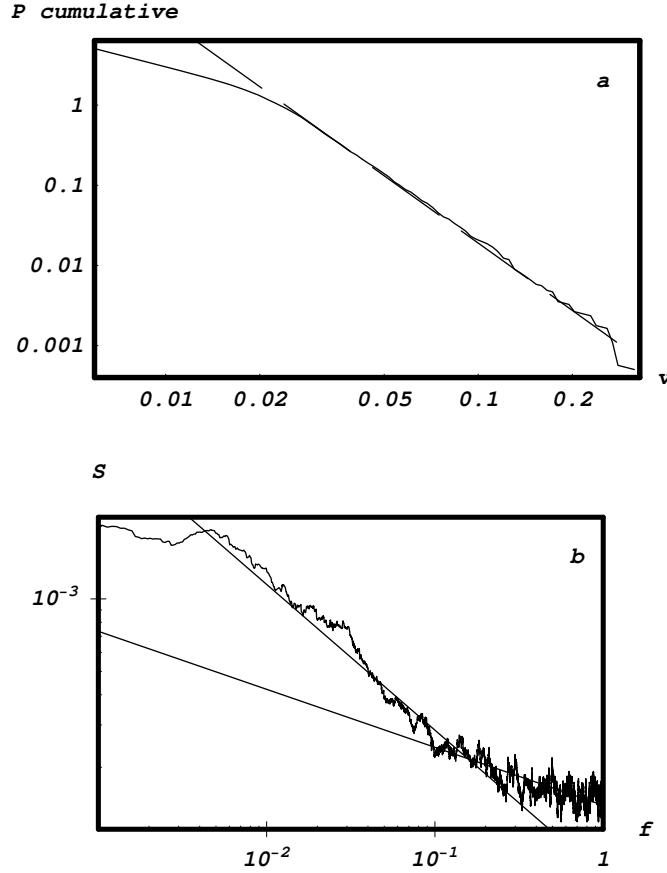
Figure 5. (a) Cumulative probability distribution function of the volatility, $P_>(\overline{v})$, averaged over 10 intervals calculated from the series of $n(t)$ generated by Eqs. (13) and (20), all parameters are the same as in previous calculations. Dashed line approximates the power law $P(\overline{v}) \sim 1/\overline{v}^{2.8}$. (b) Power spectral density $S(f)$ of v calculated from FFT of the same series $n(t)$. Straight lines approximate power spectral density $S \sim 1/f^{\beta_{1,2}}$ with $\beta_1 = 0.6$ and $\beta_2 = 0.24$.

subsequent time intervals τ_d as $N(t, \tau_\mathrm{d}) = \int_t^{t+\tau_\mathrm{d}} n(t')dt'$. This enables us to generate series of return $x(t, \tau_\mathrm{d})$, of volatility $v(t, \tau_\mathrm{d}) = |x(t, \tau_\mathrm{d})|$ and of the averaged volatility $\overline{v} = \frac{1}{m} \sum_{i=1}^{i=m} v(t_i, \tau_\mathrm{d})$.

In figure 5 we demonstrate cumulative distribution of $\overline{v}$ and the power spectral density of $v(t, \tau_\mathrm{d})$ calculated from FFT. We see that proposed model enables us to catch up the main features of the volatility: the power law distribution with exponent 2.8 and power spectral density with two exponents $\beta_1 = 0.6$ and $\beta_2 = 0.24$. This is in a good agreement with the empirical data [19, 23].

6. Conclusions

Earlier proposed stochastic point process model [14, 18] as a possible model of trading activity in the financial markets has to be elaborated. First of all, we define that

the long-range memory fluctuations of trading activity in financial markets may be considered as background stochastic process responsible for the fractal properties of other financial variables. Waiting time in the sequence of trades more likely is double stochastic process, i.e., Poisson process with the stochastic rate defined as a stand-alone stochastic variable. We consider the stochastic rate as continuous one and model it by the stochastic differential equation, exhibiting long-range memory properties. We reconsider previous stochastic point process as continuous process and propose the related nonlinear stochastic differential equation with the same statistical properties [13]. One more elaboration of the model is needed to build up the stochastic process with the statistical properties similar to the empirically defined properties of trading activity in the financial markets. We combine the market response function to the noise as consisting of two different powers: one responsible for the probability distribution function and another responsible for the power spectral density. The proposed new form of the continuous stochastic differential equation enables us to reproduce the main statistical properties of the trading activity and waiting time, observed in the financial markets. More precise model definition enables us to reproduce power spectral density with two different scaling exponents. This provides an evidence that the market behavior is dependant on the level of activity and two stages: calm and excited must be considered. We proposed a very simple model to reproduce the statistical properties of return and volatility. More sophisticated approach has to be elaborated to account for the leverage effect and other specific features of the market.

Acknowledgements

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